Gradient Discretization of Hybrid Dimensional Darcy Flows in Fractured Porous Media

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Abstract This article deals with the discretization of hybrid dimensional model of Darcy flow in fractured porous media. These models couple the flow in the fractures represented as the surfaces of codimension one with the flow in the surrounding matrix. The convergence analysis is carried out in the framework of Gradient schemes which accounts for a large family of conforming and nonconforming discretizations. The Vertex Approximate Gradient (VAG) scheme and the Hybrid Finite Volume (HFV) scheme are applied to such models and are shown to verify the Gradient scheme framework. Our theoretical results are confirmed by a few numerical experiments performed both on tetrahedral and hexahedral meshes in heterogeneous isotropic and anisotropic media.

1 Hybrid Dimensional Darcy Flow in Fractured Porous Media

Let Ω denote a bounded polyhedral domain of $\mathbb{R}^d$, $d = 2,3$. We consider the asymptotic model introduced in [1] where fractures are represented as interfaces of codimension 1. Let $\overline{\Gamma} = \bigcup_{i\in I} \overline{\Gamma}_i$ denotes the network of fractures $\Gamma_i \subset \Omega$, $i \in I$, such that each Γ_i is a planar polygonal simply connected open domain. It is assumed that the angles of Γ_i are strictly lower than 2π and that $\Gamma_i \cap \Gamma_j = \emptyset$ for all $i \neq j$. For all $i \in I$, let us set $\Sigma_i = \partial\Gamma_i$, $\Sigma_{i,j} = \Sigma_i \cap \Sigma_j$, $j \in I$, $\Sigma_{i,0} = \Sigma_i \cap \partial\Omega$, $\Sigma_{i,N} = \Sigma_i \setminus (\bigcup_{j\in I} \Sigma_{i,j} \cup \Sigma_{i,0})$, and $\Sigma = \bigcup_{(i,j)\in I\times I, i\neq j} \Sigma_{i,j}$. It is assumed that $\Sigma_{i,0} = \overline{\Gamma}_i \cap \partial\Omega$, and that $\bigcup_{i\in I} \Gamma_i = \Gamma \setminus \Sigma$. We

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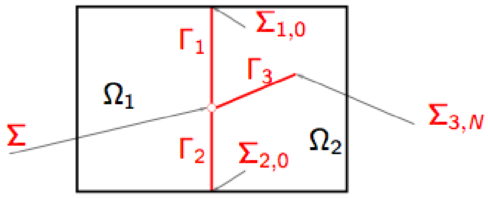

Fig. 1 Example of a 2D domain with 3 intersecting fractures and 2 connected components.

will denote by $d\tau(\mathbf{x})$ the $d-1$ dimensional Lebesgue measure on Γ. Let $H^1(\Gamma)$ denote the set of functions $v = (v_i)_{i\in I}$ such that $v_i \in H^1(\Gamma_i)$, $i \in I$ with continuous traces at the fracture intersections, and endowed with the norm $\|v\|^2_{H^1(\Gamma)} = \sum_{i\in I} \|v_i\|^2_{H^1(\Gamma_i)}$. Its subspace with vanishing traces on $\Sigma_0 = \bigcup_{i\in I} \Sigma_{i,0}$ is denoted by $H^1_{\Sigma_0}(\Gamma)$. The gradient operator from $H^1(\Omega)$ to $L^2(\Omega)^d$ is denoted by ∇, and the tangential gradient from $H^1(\Gamma)$ to $L^2(\Gamma)^{d-1}$ by ∇_τ. Let us also consider the trace operator γ from $H^1(\Omega)$ to $L^2(\Gamma)$. The function spaces used in the variational formulation of the hybrid dimensional Darcy flow model are defined by

$$V = \{v \in H^1(\Omega),\, \gamma v \in H^1(\Gamma)\}, \text{ and its subspace } V_0 = \{v \in H^1_0(\Omega),\, \gamma v \in H^1_{\Sigma_0}(\Gamma)\}.$$

The space V_0 is endowed with the norm $\|v\|^2_{V_0} = \|\nabla v\|^2_{L^2(\Omega)^d} + \|\nabla_\tau \gamma v\|^2_{L^2(\Gamma)^{d-1}}$ and the space V with the norm $\|v\|^2_V = \|v\|^2_{V_0} + \|v\|^2_{L^2(\Omega)}$. Let $\Omega_\alpha, \alpha \in \Xi$ denote the connected components of $\Omega \setminus \overline{\Gamma}$, and let us define the space $H_{\text{div}}(\Omega \setminus \overline{\Gamma}) = \{\mathbf{q}_m = (\mathbf{q}_{m,\alpha})_{\alpha\in\Xi} \,|\, \mathbf{q}_{m,\alpha} \in H_{\text{div}}(\Omega_\alpha)\}$. For all $i \in I$, we can define the two sides $\pm$ of the fracture Γ_i and the corresponding unit normal vector $n_i^\pm$ at Γ_i outward to the sides $\pm$. For all $\mathbf{q}_m \in H_{\text{div}}(\Omega \setminus \overline{\Gamma})$, let $\mathbf{q}_m^\pm \cdot n_i^\pm|_{\Gamma_i}$ denote the two normal traces at the fracture Γ_i and let us define the jump operator $H_{\text{div}}(\Omega \setminus \overline{\Gamma}) \to \big(H^{1/2}_{00}(\Gamma_i)\big)'$ by $[\![\mathbf{q}_m \cdot \mathbf{n}_i]\!] = \mathbf{q}_m^+ \cdot n_i^+|_{\Gamma_i} + \mathbf{q}_m^- \cdot n_i^-|_{\Gamma_i}$. For all fractures Γ_i, $i \in I$, we denote by $\mathbf{n}_{\Sigma_i}$ the unit vector normal to Σ_i outward to Γ_i.

Hybrid Dimensional Darcy Flow Model: In the matrix domain $\Omega \setminus \overline{\Gamma}$ (resp. in the fracture network Γ), let us denote by $\Lambda_m(\mathbf{x})$ (resp. $\Lambda_f(\mathbf{x})$) the permeability tensor. We also denote by $d_f(\mathbf{x}), \mathbf{x} \in \Gamma$ the width of the fractures, and by $d\tau_f(\mathbf{x})$ the weighted Lebesgue $d-1$ dimensional measure on Γ defined by $d\tau_f(\mathbf{x}) = d_f(\mathbf{x})d\tau(\mathbf{x})$. We consider the source terms $h_m \in L^2(\Omega)$ (resp. $h_f \in L^2(\Gamma)$) in the matrix domain $\Omega \setminus \overline{\Gamma}$ (resp. in the fracture network Γ). The strong formulation of the model amounts to find $u \in V_0$, $(\mathbf{q}_m, \mathbf{q}_f) \in W(\Omega, \Gamma)$ such that

$$\left\{\begin{array}{rll} \text{div}(\mathbf{q}_{m,\alpha}) = h_m & \text{on } \Omega_\alpha, \alpha \in \Xi, \\ \mathbf{q}_{m,\alpha} = -\Lambda_m \nabla u & \text{on } \Omega_\alpha, \alpha \in \Xi, \\ \text{div}_\tau(\mathbf{q}_{f,i}) - [\![\mathbf{q}_m \cdot \mathbf{n}_i]\!] = d_f h_f & \text{on } \Gamma_i, i \in I, \\ \mathbf{q}_{f,i} = -d_f\, \Lambda_f \nabla_\tau \gamma u & \text{on } \Gamma_i, i \in I, \end{array}\right. \tag{1}$$

where the function space $W(\Omega,\Gamma)$ is defined by

$$
\begin{aligned}
W(\Omega,\Gamma) = \{ \ & \mathbf{q}_m = (\mathbf{q}_{m,\alpha})_{\alpha\in\Xi}, \mathbf{q}_f = (\mathbf{q}_{f,i})_{i\in I} \,|\, \mathbf{q}_m \in H_{\mathrm{div}}(\Omega\setminus\overline{\Gamma}),\\
& \mathbf{q}_{f,i} \in L^2(\Gamma_i)^{d-1}, r_{f,i} = \mathrm{div}_\tau(\mathbf{q}_{f,i}) - [\![\mathbf{q}_m\cdot\mathbf{n}_i]\!] \in L^2(\Gamma_i), i\in I,\\
& \sum_{\alpha\in\Xi}\int_{\Omega_\alpha} (\mathbf{q}_{m,\alpha}\cdot\nabla v + \mathrm{div}(\mathbf{q}_{m,\alpha})v)d\mathbf{x}\\
& +\sum_{i\in I}\int_{\Gamma_i}(\mathbf{q}_{f,i}\cdot\nabla_\tau\gamma v + r_{f,i}\gamma v)d\tau = 0 \text{ for all } v\in V_0\}.
\end{aligned}
$$

The last condition corresponds to impose in a weak sense that $\sum_{i\in I}\mathbf{q}_{f,i}\cdot\mathbf{n}_{\Sigma_i} = 0$ on Σ and $\mathbf{q}_{f,i}\cdot\mathbf{n}_{\Sigma_i} = 0$ on $\Sigma_{i,N}, i\in I$.

In variational form, (1) amounts to find $u\in V_0$ such that for all $v\in V_0$:

$$
\left\{
\begin{aligned}
& \int_\Omega \Lambda_m(\mathbf{x})\nabla u(\mathbf{x})\cdot\nabla v(\mathbf{x})d\mathbf{x} + \int_\Gamma \Lambda_f(\mathbf{x})\nabla_\tau\gamma u(\mathbf{x})\cdot\nabla_\tau\gamma v(\mathbf{x})d\tau_f(\mathbf{x})\\
& -\int_\Omega h_m(\mathbf{x})v(\mathbf{x})d\mathbf{x} - \int_\Gamma h_f(\mathbf{x})\gamma v(\mathbf{x})d\tau_f(\mathbf{x}) = 0.
\end{aligned}
\right. \tag{2}
$$

Proposition 1. *From the Lax-Milgram theorem, the variational problem* (2) *has a unique solution $u\in V_0$ which satisfies the a priori estimate* $\|u\|_V \le C\Big(\|h_m\|_{L^2(\Omega)} + \|h_f\|_{L^2(\Gamma)}\Big)$, *with C depending only on Ω, Γ, Λ_m, Λ_f, d_f. In addition $(\mathbf{q}_m = -\Lambda_m\nabla u, \mathbf{q}_f = -d_f\Lambda_f\nabla_\tau\gamma u)$ belongs to $W(\Omega,\Gamma)$.*

2 Gradient Discretization

A gradient discretization $\mathscr{D}$ of (2) is defined by a vector space of degrees of freedom $X_{\mathscr{D}}$, its subspace associated with homogeneous Dirichlet boundary conditions $X^0_{\mathscr{D}}$, and the following set of linear operators:

- Gradient operator on the matrix domain: $\nabla_{\mathscr{D}_m} : X_{\mathscr{D}} \to L^2(\Omega)^d$
- Gradient operator on the fracture network: $\nabla_{\mathscr{D}_f} : X_{\mathscr{D}} \to L^2(\Gamma)^{d-1}$
- A function reconstruction operator on the matrix domain: $\Pi_{\mathscr{D}_m} : X_{\mathscr{D}} \to L^2(\Omega)$
- A function reconstruction operator on the fracture network: $\Pi_{\mathscr{D}_f} : X_{\mathscr{D}} \to L^2(\Gamma)$.

$X_{\mathscr{D}}$ is endowed with the semi-norm $\|v_{\mathscr{D}}\|^2_{\mathscr{D}} = \|\nabla_{\mathscr{D}_m} v_{\mathscr{D}}\|^2_{L^2(\Omega)^d} + \|\nabla_{\mathscr{D}_f} v_{\mathscr{D}}\|^2_{L^2(\Gamma)^{d-1}}$ which is assumed to define a norm on $X^0_{\mathscr{D}}$. Next, we define the coercivity, consistency, limit conformity and compactness properties of the gradient discretization.

Coercivity: There exists $C_{\mathscr{D}} \ge 0$ such that for all $v\in X^0_{\mathscr{D}}$ one has

$$\|\Pi_{\mathscr{D}_m} v_{\mathscr{D}}\|_{L^2(\Omega)} + \|\Pi_{\mathscr{D}_f} v_{\mathscr{D}}\|_{L^2(\Gamma)} \le C_{\mathscr{D}}\|v_{\mathscr{D}}\|_{\mathscr{D}}.$$

Consistency: Let $u\in V_0$, and let us define

$$\mathscr{S}_{\mathscr{D}}(u) = \inf_{v_{\mathscr{D}} \in X^0_{\mathscr{D}}} \Big(\|\nabla_{\mathscr{D}_m} v_{\mathscr{D}} - \nabla u\|_{L^2(\Omega)^d} + \|\nabla_{\mathscr{D}_f} v_{\mathscr{D}} - \nabla_\tau \gamma u\|_{L^2(\Gamma)^{d-1}} \\ + \|\Pi_{\mathscr{D}_m} v_{\mathscr{D}} - u\|_{L^2(\Omega)} + \|\Pi_{\mathscr{D}_f} v_{\mathscr{D}} - \gamma u\|_{L^2(\Gamma)} \Big)$$

Then, a sequence of gradient discretizations $(\mathscr{D}^l)_{l\in\mathbb{N}}$ is said to be consistent if for all $u \in V_0$ one has $\lim_{l\to+\infty} \mathscr{S}_{\mathscr{D}^l}(u) = 0$.

Limit Conformity: For all $(\mathbf{q}_m, \mathbf{q}_f) \in W(\Omega, \Gamma)$, we define

$$\mathscr{W}_{\mathscr{D}}(\mathbf{q}_m, \mathbf{q}_f) = \sup_{0 \neq v_{\mathscr{D}} \in X^0_{\mathscr{D}}} \frac{1}{\|v_{\mathscr{D}}\|_{\mathscr{D}}} \Big(\sum_{\alpha\in\Xi} \int_{\Omega_\alpha} (\nabla_{\mathscr{D}_m} v_{\mathscr{D}} \cdot \mathbf{q}_{m,\alpha} + (\Pi_{\mathscr{D}_m} v_{\mathscr{D}}) \mathrm{div}(\mathbf{q}_{m,\alpha}))(\mathbf{x}) d\mathbf{x} \\ + \sum_{i\in I} \int_{\Gamma_i} (\nabla_{\mathscr{D}_f} v_{\mathscr{D}} \cdot \mathbf{q}_f + \Pi_{\mathscr{D}_f} v_{\mathscr{D}} (\mathrm{div}_{\tau_i}(\mathbf{q}_{f,i}) - [\![\mathbf{q}_m \cdot \mathbf{n}_i]\!]))(\mathbf{x}) d\tau(\mathbf{x}) \Big). \tag{3}$$

Then, a sequence of gradient discretizations $(\mathscr{D}^l)_{l\in\mathbb{N}}$ is said to be limit conforming if for all $(\mathbf{q}_m, \mathbf{q}_f) \in W(\Omega, \Gamma)$ one has $\lim_{l\to+\infty} \mathscr{W}_{\mathscr{D}^l}(\mathbf{q}_m, \mathbf{q}_f) = 0$.

Compactness: A sequence of gradient discretizations $(\mathscr{D}^l)_{l\in\mathbb{N}}$ is said to be compact if for all sequences $v_{\mathscr{D}^l} \in X^0_{\mathscr{D}^l}$, $l \in \mathbb{N}$ such that there exists $C > 0$ with $\|v_{\mathscr{D}^l}\|_{\mathscr{D}^l} \leq C$ for all $l \in \mathbb{N}$, then there exist $u_m \in L^2(\Omega)$ and $u_f \in L^2(\Gamma)$ with

$$\lim_{l\to+\infty} \|\Pi_{\mathscr{D}^l_m} v_{\mathscr{D}^l} - u_m\|_{L^2(\Omega)} = 0 \text{ and } \lim_{l\to+\infty} \|\Pi_{\mathscr{D}^l_f} v_{\mathscr{D}^l} - u_f\|_{L^2(\Gamma)} = 0.$$

The discretization of (2) using the Gradient Scheme framework is defined by: find $u \in X^0_{\mathscr{D}}$ such that for all $v_{\mathscr{D}} \in X^0_{\mathscr{D}}$:

$$\left\{ \begin{array}{l} \displaystyle\int_\Omega \Lambda_m(\mathbf{x}) \nabla_{\mathscr{D}_m} u_{\mathscr{D}}(\mathbf{x}) \cdot \nabla_{\mathscr{D}_m} v_{\mathscr{D}}(\mathbf{x}) d\mathbf{x} + \int_\Gamma \Lambda_f(\mathbf{x}) \nabla_{\mathscr{D}_f} u_{\mathscr{D}}(\mathbf{x}) \cdot \nabla_{\mathscr{D}_f} v_{\mathscr{D}}(\mathbf{x}) d\tau_f(\mathbf{x}) \\ \displaystyle - \int_\Omega h_m(\mathbf{x}) \Pi_{\mathscr{D}_m} v_{\mathscr{D}}(\mathbf{x}) d\mathbf{x} - \int_\Gamma h_f(\mathbf{x}) \Pi_{\mathscr{D}_f} v_{\mathscr{D}}(\mathbf{x}) d\tau_f(\mathbf{x}) = 0. \end{array} \right. \tag{4}$$

Proposition 2. *Let $\mathscr{D}$ be a gradient discretization of* (2) *assumed to be coercive. Then* (4) *has a unique solution $u_{\mathscr{D}} \in X^0_{\mathscr{D}}$ satisfying the a priori estimate $\|u_{\mathscr{D}}\|_{\mathscr{D}} \leq C\Big(\|h_m\|_{L^2(\Omega)} + \|h_f\|_{L^2(\Gamma)}\Big)$ with C depending only on $C_{\mathscr{D}}$, Λ_m, Λ_f, d_f.*

Proposition 3. Error Estimates*. Let $u \in V_0$, $(\mathbf{q}_m, \mathbf{q}_f) \in W(\Omega, \Gamma)$ be the solution of* (2)*. Let $\mathscr{D}$ be a gradient discretization of* (2) *assumed to be coercive, and let $u_{\mathscr{D}} \in X^0_{\mathscr{D}}$ be the solution of* (4)*. Then, there exist C_1, C_2, C_3, C_4 depending only on $C_{\mathscr{D}}$, Λ_m, Λ_f, d_f such that one has the following error estimates:*

$$\left\{ \begin{array}{l} \|\nabla u - \nabla_{\mathscr{D}_m} u_{\mathscr{D}}\|_{L^2(\Omega)^d} + \|\nabla_\tau \gamma u - \nabla_{\mathscr{D}_f} u_{\mathscr{D}}\|_{L^2(\Gamma)^{d-1}} \leq C_1 \mathscr{S}_{\mathscr{D}}(u) + C_2 \mathscr{W}(\mathbf{q}_m, \mathbf{q}_f), \\ \|\Pi_{\mathscr{D}_m} u_{\mathscr{D}} - u\|_{L^2(\Omega)} + \|\Pi_{\mathscr{D}_f} u_{\mathscr{D}} - \gamma u\|_{L^2(\Gamma)} \leq C_3 \mathscr{S}_{\mathscr{D}}(u) + C_4 \mathscr{W}(\mathbf{q}_m, \mathbf{q}_f). \end{array} \right.$$

3 Two Examples of Gradient Discretizations of Hybrid Dimensional Models

In the spirit of [3], we consider generalized polyhedral meshes of Ω. Let $\mathscr{M}$ be the set of cells that are disjoint open polyhedral subsets of Ω such that $\bigcup_{K\in\mathscr{M}} \overline{K} = \overline{\Omega}$. For all $K \in \mathscr{M}$, $\mathbf{x}_K$ denotes the so-called "centre" of the cell K under the assumption that K is star-shaped with respect to $\mathbf{x}_K$. We then denote by $\mathscr{F}_K$ the set of interfaces of non zero $d-1$ dimensional measure among the interior faces $\overline{K}\cap\overline{L}$, $L \in \mathscr{M}$, and the boundary interface $\overline{K}\cap\partial\Omega$, which possibly splits in several boundary faces. Let us denote by $\mathscr{F} = \bigcup_{K\in\mathscr{M}} \mathscr{F}_K$ the set of all faces of the mesh. The term "generalized polyhedral mesh" means that the faces are not assumed to be planar. For $\sigma \in \mathscr{F}$, let $\mathscr{E}_\sigma$ be the set of interfaces of non zero $d-2$ dimensional measure among the interfaces $\sigma\cap\sigma'$, $\sigma' \in \mathscr{F}$. Then, we denote by $\mathscr{E} = \bigcup_{\sigma\in\mathscr{F}} \mathscr{E}_\sigma$ the set of all edges of the mesh. Let $\mathscr{V}_\sigma = \bigcup_{e,e'\in\mathscr{E}_\sigma, e\neq e'} \left(e\cap e'\right)$ be the set of vertices of σ, for each $K \in \mathscr{M}$ we define $\mathscr{V}_K = \bigcup_{\sigma\in\mathscr{F}_K} \mathscr{V}_\sigma$, and we also denote by $\mathscr{V} = \bigcup_{K\in\mathscr{M}} \mathscr{V}_K$ the set of all vertices of the mesh. It is then assumed that for each face $\sigma \in \mathscr{F}$, there exists a so-called "centre" of the face $\mathbf{x}_\sigma \in \sigma \setminus \bigcup_{e\in\mathscr{E}_\sigma} e$ such that $\mathbf{x}_\sigma = \sum_{\mathbf{s}\in\mathscr{V}_\sigma} \beta_{\sigma,\mathbf{s}}\, \mathbf{x}_\mathbf{s}$, with $\sum_{\mathbf{s}\in\mathscr{V}_\sigma} \beta_{\sigma,\mathbf{s}} = 1$, and $\beta_{\sigma,\mathbf{s}} \geq 0$ for all $\mathbf{s} \in \mathscr{V}_\sigma$; moreover the face σ is assumed to match with the union of the triangles $T_{\sigma,e}$ defined by the face centre $\mathbf{x}_\sigma$ and each edge $e \in \mathscr{E}_\sigma$. The mesh is also supposed to be conforming w.r.t. the fracture network Γ in the sense that for all $i \in I$ there exist the subsets $\mathscr{F}_{\Gamma_i}$ of $\mathscr{F}$ such that $\overline{\Gamma}_i = \bigcup_{\sigma\in\mathscr{F}_{\Gamma_i}} \sigma$. We will denote by $\mathscr{F}_\Gamma$ the set of fracture faces $\bigcup_{i\in I} \mathscr{F}_{\Gamma_i}$.
The discretization of the hybrid dimensional Darcy flow model with continuous pressures has been the object of several works such as [6] using a cell centred MultiPoint Flux Approximation scheme, [1] using a Mixed Finite Element (MFE) method, and [5] using a Control Volume Finite Element Method (CVFE). The MFE method, as well as some CVFE and MPFA schemes on e.g. tetrahedral meshes can be shown to be gradient discretizations. In the following we propose to apply the VAG and HFV schemes.
Vertex Approximate Gradient Discretization: The VAG discretization has been introduced in [3] for diffusive problems on heterogeneous anisotropic media. Its extension to the hybrid dimensional two-phase Darcy flow model is presented in [2]. The scheme is based on the following vector space of degrees of freedom:

$$X_\mathscr{D} = \{u_K, u_\mathbf{s}, u_\sigma \in \mathbb{R} \text{ for all } K \in \mathscr{M}, \mathbf{s} \in \mathscr{V}, \sigma \in \mathscr{F}_\Gamma\},$$

and its subspace with homogeneous Dirichlet boundary conditions on $\partial\Omega$: $X^0_\mathscr{D} = \{u \in X_\mathscr{D} \,|\, u_\mathbf{s} = 0 \text{ for } \mathbf{s} \in \mathscr{V}_{ext}\}$ where $\mathscr{V}_{ext} = \mathscr{V} \cap \partial\Omega$ denotes the set of boundary vertices, and $\mathscr{V}_{int} = \mathscr{V} \setminus \mathscr{V}_{ext}$ denotes the set of interior vertices.

The discrete gradients in the matrix and in the fracture are defined as the usual gradient operators on the conforming space of continuous affine finite elements built upon a tetrahedral sub-mesh. In addition, the VAG discretization uses two non conforming piecewise constant reconstructions of functions from $X_\mathscr{D}$ into respectively $L^2(\Omega)$ and $L^2(\Gamma)$. In the matrix, it is such that $\pi_{\mathscr{D}_m} u(\mathbf{x})|_{\omega_{m,\nu}} = u_\nu$ where the $\omega_{m,\nu}$ for $\nu \in \mathscr{M} \cup \mathscr{V}_{int} \cup \mathscr{F}_\Gamma$ are neighbourhoods of $\mathbf{x}_\nu$ defining a partition of Ω. In the fractures, it is such that $\pi_{\mathscr{D}_f} u(\mathbf{x})|_{\omega_{f,\nu}} = u_\nu$ where the $\omega_{f,\nu}$ for $\nu \in (\mathscr{V}_\Gamma \cap \mathscr{V}_{int}) \cup \mathscr{F}_\Gamma$ are neighbourhoods of $\mathbf{x}_\nu$ defining a partition of Γ.

Hybrid Finite Volume Discretization: The Hybrid Finite Volume (HFV) scheme introduced in [4] can be extended to the hybrid dimensional Darcy flow model as follows. The faces $\sigma \in \mathscr{F}$ are assumed to be planar and $\mathbf{x}_\sigma$ is assumed to be the centre of gravity of the face σ. We also denote by $\mathbf{x}_e$ the centre of the edge $e \in \mathscr{E}$. Let $\mathscr{F}_{int} \subset \mathscr{F}$ (resp. $\mathscr{E}_{int} \subset \mathscr{E}$) denote the subset of interior faces (resp. interior edges). The vector space of degrees of freedom $X_\mathscr{D}$ is defined by

$$X_\mathscr{D} = \{u_K, u_\sigma, u_e \in \mathbb{R} \text{ for all } K \in \mathscr{M}, \sigma \in \mathscr{F}, e \in \mathscr{E}_\Gamma\},$$

where $\mathscr{E}_\Gamma \subset \mathscr{E}$ denotes the subset of edges of Γ, and its subspace $X^0_\mathscr{D}$ is such that $u_\sigma = 0$ for all $\sigma \in \mathscr{F} \setminus \mathscr{F}_{int}$ and $u_e = 0$ for all $e \in \mathscr{E}_\Gamma \setminus \mathscr{E}_{int}$. For each cell K and $u \in X_\mathscr{D}$, let us define $\nabla_K u = \frac{1}{|K|} \sum_{\sigma \in \mathscr{F}_K} |\sigma|(u_\sigma - u_K)\mathbf{n}_{K,\sigma}$, where $|K|$ is the volume of the cell K, $|\sigma|$ is the surface of the face σ, and $\mathbf{n}_{K,\sigma}$ is the unit normal vector of the face $\sigma \in \mathscr{F}_K$ outward to the cell K. The discrete gradient $\nabla_K u$ is stabilized using $\nabla_{K,\sigma} u = \nabla_K u + R_{K,\sigma}(u)\mathbf{n}_{K,\sigma}$, $\sigma \in \mathscr{F}_K$, with $R_{K,\sigma}(u) = \frac{\sqrt{d}}{d_{K,\sigma}} \Big(u_\sigma - u_K - \nabla_K u \cdot (\mathbf{x}_K - \mathbf{x}_\sigma)\Big)$, and $d_{K,\sigma} = \mathbf{n}_{K,\sigma} \cdot (\mathbf{x}_\sigma - \mathbf{x}_K)$ which leads to the definition of the matrix discrete gradient $\nabla_{\mathscr{D}_m} u(\mathbf{x}) = \nabla_{K,\sigma} u$ on K_σ for all $K \in \mathscr{M}, \sigma \in \mathscr{F}_K$, where K_σ is the cone joining the face σ to the cell centre $\mathbf{x}_K$. The fracture discrete gradient is defined similarly by $\nabla_{\mathscr{D}_f} u(\mathbf{x}) = \nabla_{\sigma,e} u$ on σ_e for all $\sigma \in \mathscr{F}_\Gamma, e \in \mathscr{E}_\sigma$, with $\nabla_{\sigma,e} u = \nabla_\sigma u + R_{\sigma,e}(u)\mathbf{n}_{\sigma,e}$, and $\nabla_\sigma u = \frac{1}{|\sigma|} \sum_{e \in \mathscr{E}_\sigma} |e|(u_e - u_\sigma)\mathbf{n}_{\sigma,e}, R_{\sigma,e}(u) = \frac{\sqrt{d-1}}{d_{\sigma,e}} \Big(u_e - u_\sigma - \nabla_\sigma u \cdot (\mathbf{x}_\sigma - \mathbf{x}_e)\Big)$, where $\mathbf{n}_{\sigma,e}$ is the unit normal vector to the edge e in the tangent plane of the face σ and outward to the face σ, $d_{\sigma,e} = \mathbf{n}_{\sigma,e} \cdot (\mathbf{x}_e - \mathbf{x}_\sigma)$, and σ_e is the triangle of base e and vertex $\mathbf{x}_\sigma$. The function reconstruction operators are piecewise constant on a partition of the cells and of the fracture faces. These partitions are respectively denoted, for all $K \in \mathscr{M}$, by $K = \omega_K \cup \Big(\bigcup_{\sigma \in \mathscr{F}_K \cap \mathscr{F}_{int}} \omega_{K,\sigma}\Big)$, and, for all $\sigma \in \mathscr{F}_\Gamma$, by $\sigma = \Sigma_\sigma \cup \Big(\bigcup_{e \in \mathscr{E}_\sigma \cap \mathscr{E}_{int}} \Sigma_{\sigma,e}\Big)$. Then, the function reconstruction operators are defined by $\Pi_{\mathscr{D}_m} u(\mathbf{x}) = \begin{cases} u_K \text{ for all } \mathbf{x} \in \omega_K,\ K \in \mathscr{M}, \\ u_\sigma \text{ for all } \mathbf{x} \in \omega_{K,\sigma},\ \sigma \in \mathscr{F}_K \cap \mathscr{F}_{int},\ K \in \mathscr{M}, \end{cases}$ and $\Pi_{\mathscr{D}_f} u(\mathbf{x}) = \begin{cases} u_\sigma \text{ for all } \mathbf{x} \in \Sigma_\sigma,\ \sigma \in \mathscr{F}_\Gamma, \\ u_e \text{ for all } \mathbf{x} \in \Sigma_{\sigma,e},\ e \in \mathscr{E}_\sigma \cap \mathscr{E}_{int},\ \sigma \in \mathscr{F}_\Gamma. \end{cases}$

We can show the following proposition which can be proven using a lemma stating the density of smooth function subspaces in the spaces V, V_0, and $W(\Omega, \Gamma)$.

Proposition 4. *Let us consider a family of meshes $\mathscr{M}^{(m)}$, $m \in \mathbb{N}$ as defined above. It is assumed that the family of tetrahedral submeshes of $\mathscr{M}^{(m)}$ is shape regular, that the cardinal of $\mathscr{V}_K$ is uniformly bounded for all $K \in \mathscr{M}^{(m)}$, and all $m \in \mathbb{N}$, and that the maximum diameter $h^{(m)}$ of the cells $K \in \mathscr{M}^{(m)}$ tends to zero with $m \to +\infty$. In addition, in the case of the HFV scheme, the faces are assumed to be planar. Then, the VAG and HFV discretizations are coercive, consistent, limit conforming and compact gradient discretizations of the hybrid dimensional Darcy flow model.*

4 Numerical Experiments

Let $\Omega = (0,1)^3$ and consider the 2 planar fractures defined by $x = 0.5$ and $y = 0.5$ and splitting Ω into the four subdomains Ω_α, $\alpha = 1, \cdots, 4$ corresponding respectively to $\{x < 0.5, y < 0.5\}$, $\{x > 0.5, y < 0.5\}$, $\{x > 0.5, y > 0.5\}$ and $\{x < 0.5, y > 0.5\}$. In the fractures, we set $\Lambda_f(\mathbf{x}) = 100\, I$ and $d_f(\mathbf{x}) = 0.01$. In the matrix, the permeability tensor $\Lambda_m(\mathbf{x})$ is fixed to $\Lambda_{m,\alpha}$ on each subdomain Ω_α, $\alpha = 1, \cdots, 4$ with two choices of the subdomain permeabilities. The first choice considers isotropic heterogeneous permeabilities setting $\Lambda_{m,\alpha} = \lambda_\alpha\, I$ with $\lambda_1 = 1$, $\lambda_2 = 0.1$, $\lambda_3 = 0.01$, $\lambda_4 = 10$. The second choice defines anisotropic heterogeneous permeabilities by

$$\Lambda_{m,1} = \begin{pmatrix} a_1 & b_1 & 0 \\ b_1 & c_1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \Lambda_{m,2} = \begin{pmatrix} a_2 & 0 & b_2 \\ 0 & \lambda & 0 \\ b_2 & 0 & c_2 \end{pmatrix}, \Lambda_{m,3} = \begin{pmatrix} a_3 & b_3 & 0 \\ b_3 & c_3 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \Lambda_{m,4} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & a_4 & b_4 \\ 0 & b_4 & c_4 \end{pmatrix},$$

with $a_\alpha = \cos^2\beta_\alpha + \omega \sin^2\beta_\alpha$, $b_\alpha = (1-\omega)\cos\beta_\alpha \sin\beta_\alpha$, $c_\alpha = \omega\cos^2\beta_\alpha + \sin^2\beta_\alpha$, $\lambda = 0.01$, $\beta_1 = \frac{\pi}{6}$, $\beta_2 = -\frac{\pi}{6}$, $\beta_3 = 0$, $\beta_4 = \frac{\pi}{6}$ and $\omega = 0.01$. For each subdomain let us define $t_1(\mathbf{x}) = y - x + z$, $t_2(\mathbf{x}) = x + y + z - 1$, $t_3(\mathbf{x}) = x - y + z$ and $t_4(\mathbf{x}) = 1 - x - y + z$. It can be checked that the function $u(\mathbf{x}) = e^{\cos(t_\alpha(\mathbf{x}))}$, $\mathbf{x} \in \Omega_\alpha$, $\alpha = 1, \cdots, 4$, belongs to V and is such that $\mathbf{q}_m(\mathbf{x}) = -\Lambda_m \nabla u(\mathbf{x})$, $\mathbf{q}_f(\mathbf{x}) = -d_f \Lambda_f \nabla_\tau \gamma u(\mathbf{x})$ belongs to $W(\Omega, \Gamma)$. It will be used as exact solution of (1) with ad-hoc right hand sides and Dirichlet boundary conditions on $\partial\Omega$. For the numerical solutions, three different families of meshes are considered: uniform Cartesian meshes, a random perturbation of the previous Cartesian meshes, and tetrahedral meshes generated by TetGen. To assess the error estimates of Proposition 3, we have computed the sum of the relative L_2 norms of the errors in the matrix and in the fractures, both for the function and for the gradient reconstructions. As exhibited in Figure 2, the expected first orders of convergence are obtained both for the function reconstructions and the gradient reconstructions with observed superconvergence of order 2 for Cartesian meshes. We note that the HFV scheme seems to be less robust than the VAG scheme with respect to anisotropy. Also, as expected on tetrahedral meshes, the CPU time of the computation of the HFV solution is much larger of a factor around 10 than the CPU time obtained with the VAG scheme using for both schemes a GMRES solver preconditioned by ILUT.

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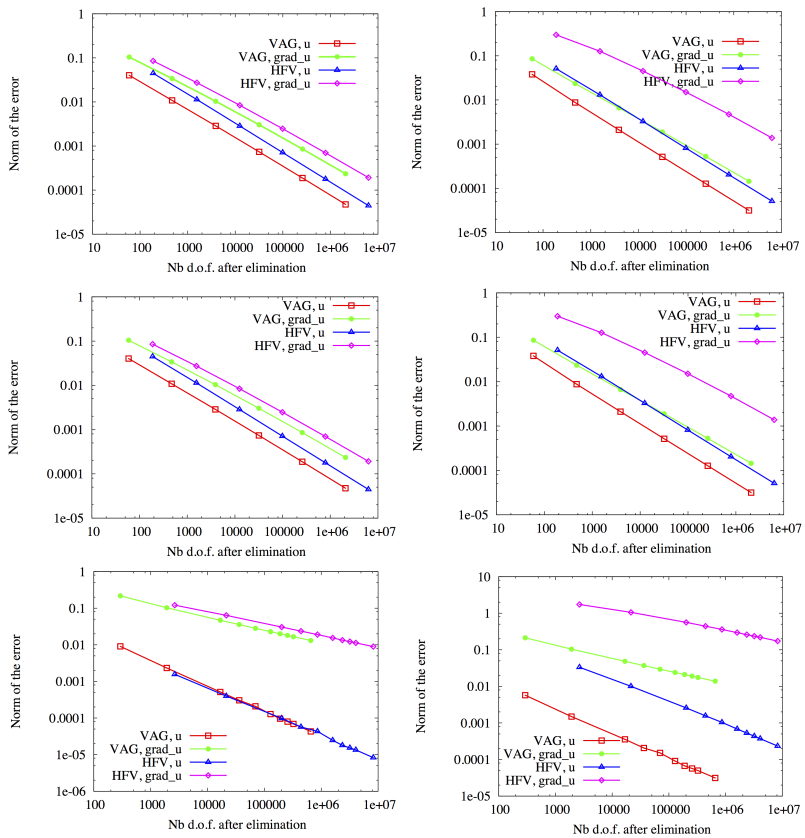

Fig. 2 For the 3 families of meshes (top: Cartesian meshes, middle: randomly perturbated Cartesian meshes, and bottom: tetrahedral meshes), and for the isotropic (left) and anisotropic (right) test cases: sum of the relative L_2 norm of the error in the matrix and in the fracture for the function and its gradients reconstructions and for both the VAG and HFV schemes function of the number of d.o.f. after elimination of the cell and Dirichlet unknowns.

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